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STABILITY AND SPECIAL METRICS FOR COMPLEX VECTOR BUNDLES WITH GLOBAL SECTIONS

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Abstract

In this paper, we study one kind of vortex equations on complex vector bundles over almost Hermitian manifolds and prove a Hitchin-Kobayashi type correspondence relating the existence of solutions of these vortex equations to a certain stability condition.

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1 Introduction

Given a holomorphic vector bundle over a compact Kähler manifold M, the relation between the existence of Hermitian-Einstein metrics and the stability condition is by now well understood due to the work of Narasimhan-Seshadri, Kobayashi, Donaldson, Siu, Uhlenbeck-Yau and others ([13], [10], [11], [4], [15], [16]). The generalization to Higgs bundles can be found in [7] and [14]. Different to the Higgs bundles, Bradlow [1], [2] considers holomorphic vector bundles on which additional data in the form of a prescribed holomorphic global section is given. Bradlow investigated the following vortex equation

$$\Lambda F_H - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*H} + \lambda \frac{\sqrt{-1}}{2} Id = 0, \tag{1.1}$$

Here F_H is the curvature of the metric connection determined by $\bar{\partial}_E$ and a Hermitian metric H, ϕ is a holomorphic section of E, ϕ^{*H} is the adjoint of ϕ with respect to metric H, and λ is a real number. This equation generalizes the Hermitian-Yang-Mills equation (which is recovered by taking $\phi = 0$) and is the analgue of the classical Vortex equation over R^2. In [2], Bradlow had proved the equivalence between the existence of solutions of the above vortex equations and the ϕ-stability of holomorphic bundle by minimizing the so-called Donaldson functional.

Recently, de Bartolomeis and Tian ([3]) investigated the stability of complex vector bundles over almost complex manifolds, they introduced the concept of bundle almost structure (bacs) J on principal bundles, defined J-stable complex vector bundles, and proved the existence of Hermitian-Einstein metrics on J-stable complex vector bundle over compact almost Hermitian regularized manifold. Inspired by this, we want to consider the vortex equations on complex vector bundle over almost Hermitian manifolds.

Let (M, J_M, g) be a compact m-dimensional almost Hermitian manifold. Given a complex vector bundle $(E, \hat{J})$ of rank r over M, we consider the principal $GL(r, C)$-bundle $C(E)$ of complex linear frames on E, and assign a bundle almost complex structure (bacs) J on $C(E)$ (which we will introduce in section two). By a proposition in [3](proposition 1.3), we can see that bacs on $C(E)$ are in one-to-one correspondence with linear differential operators

$$\bar{\partial}_E : \wedge^{p,q}(E) \longrightarrow \wedge^{p,q+1}(E)$$

satisfying $\bar{\partial}$-Leibnitz rule. We denote the set of the above differential operators by $\hat{H}(E)$. When M is a complex manifolds, we usually consider a holomorphic vector bundle E over M, and we can define partial differentiation in the $(0,1)$ direction in a natural way, i.e. the $(0,1)$ derivative of a local holomorphic section of E is defined to be zero and the $(0,1)$ derivative of any smooth section is defined by expressing it in terms of a local holomorphic basis and using the leibniz rule of differentiating products. There is no natural way to define partial differentiation in the $(0,1)$ direction when M equipped with not necessarily integrable almost complex structure, this is the reason why we should assign a (bacs) on $C(E)$.

We will assume that the complex vector bundle E has a fixed bundle almost complex structure (bacs) J, and assume further that E has a nontrivial J-holomorphic global section ϕ, i.e., ϕ satisfies

$$\bar{\partial}_E \phi = 0 \tag{1.2}$$

where $\bar{\partial}_E$ is the element of $\hat{H}(E)$ correspondence with the fixed bacs J. Let H be a Hermitian metric on E, by [3], we know there exists a unique type $(1,0)$ Hermitian connection which is called the canonical Hermitian connection ω_H. Usually, F_H denotes the curvature form of the canonical Hermitian connection. We now denote the Kähler form of the base manifold by η, and let $\Lambda : \Omega_M^{1,1} \to \Omega_M^0$ be the contraction $\Lambda(\theta) = (\theta, \eta)$. Our goal is to understand the necessary and sufficient conditions for the existence of solutions of the vortex equation (1.1) on complex vector bundle E. Our main result is the following.

Main theorem *Assume that E is a complex vector bundle over a compact almost Hermitian regularized manifold M (i.e., whose Kähler form η satisfies $\partial\bar{\partial}\eta^{m-1} = 0$), and assign a bundle almost complex structure (bacs) J on the principal $GL(r, C)$-bundle $C(E)$. Let ϕ be a nontrivial J-holomorphic section of E and suppose that (E, J, ϕ) is (J, ϕ, λ)-stable for a real number λ. Then there exists a smooth Hermitian metric H such that the vortex equation (1.1) is satisfied, i.e.*

$$\sqrt{-1}\Lambda F_H^{1,1} + \frac{1}{2}\phi \otimes \phi^{*H} - \frac{1}{2}\lambda Id = 0.$$

The plan of this paper is the following: in Section 2, we discuss the notion of bundle almost complex structure (bacs), and give some estimates which will be used in the following sections; in section 3, we give the definition of (J, ϕ, λ)-stability, and prove that (J, ϕ, λ)-stability is a necessary condition for the existence of Hermitian metrics satisfying vortex equation (1.1); in section 4, we give the proof of our main theorem. The proof of main theorem is by using the heat equation method, and adapting the techniques which already appear in the literature on the Hermitian Yang-Mills equation ([5], [14], [3]). Let H_0 be a fixed Hermitian metric on E, consider the evolution equation

$$H^{-1}\frac{d}{dt}H = -2(\sqrt{-1}\Lambda F_H^{1,1} + \frac{1}{2}\phi \otimes \phi^{*H} - \frac{1}{2}\lambda Id). \tag{1.3}$$

We prove that this evolution equation has a long time solution for any initial metric H_0, and show that the solution converges to a Hermitian metric satisfying the vortex equation (1.1), unless a flag of weakly J-holomorphic sub-bundles is produced, thanks to the regularity results in [3], we can show that there must exist one of which contradicts the stability assumption.

2 Preliminary Results

Let (M, J_M) be an m-dimensional almost complex manifold. A complex vector bundle $(E, \hat{J})$ of (complex) rank r over M is a real vector bundle E of rank 2r equipped with a section $\hat{J}$ of $End(E)$ such that $\hat{J}^2 = -Id_E$. We denote the principal $GL(r, C)$-bundle of complex linear frames on E by $C(E)$, thus E can also be seen as an associate bundle with standard fibre C^r. Firstly, we will introduce the notion of bundle almost complex structure which has been investigated by de Bartolomeis and Tian in [3].

Definition 2.1 *A bundle almost complex structure (bacs) on $C(E)$ is an almost complex structure J on $C(E)$ such that: (1), the bundle projection $\pi : C(E) \to M$ is (J, J_M)-holomorphic; (2), J induces the standard integrable almost complex structure J_S on the fibres; (3), $GL(r, C)$ acts J-holomorphically on $C(E)$.*

$B(C(E))$ will denote the set of bacs on $C(E)$. We can define

$$T^{p,q}(C(E)) = L^{-1}(\wedge^{p,q}(E)), \tag{2.1}$$

where $L : T^*(C(E)) \to \wedge^*(E)$ is the standard isomorphism between tensorial C^r-valued forms on $C(E)$ and E-valued forms on M([9]), therefore we have

$$T^n(C(E)) = \otimes_{p+q=n} T^{p,q}(C(E)). \tag{2.2}$$

It is easy to check that, if a bacs is assigned on $C(E)$, then (2.2) corresponds precisely to the induced decomposition.

Let $\hat{H}(C(E))$ be the set of all linear differential operators

$$\bar{\partial}_{C(E)} : T^{p,q}(C(E)) \to T^{p,q+1}(C(E))$$

satisfying the following $\bar{\partial}$-Leibnitz rule: for every $f \in C^\infty(M)$, $\alpha \in T^{p,q}(C(E))$

$$\bar{\partial}_{C(E)}\pi^*(f)\alpha = \pi^*(\bar{\partial}_M f) \wedge \alpha + \pi^*(f)\bar{\partial}_{C(E)}\alpha.$$

one can check that the map $J \mapsto \bar{\partial}_J$ is a bijection between $B(C(E))$ and $\hat{H}(C(E))$ ([3], proposition 1.3). On the other hand, $\hat{H}(C(E))$ is also in one-to-one correspondence with the set $\hat{H}(E)$ of linear differential operators $\bar{\partial}_E : \wedge^{p,q}(E) \to \wedge^{p,q+1}(E)$, satisfying the following $\bar{\partial}$-Leibnitz rule: for every $f \in C^\infty(M)$, $\alpha \in \wedge^{p,q}(E)$

$$\bar{\partial}_E\pi^*(f)\alpha = \bar{\partial}_M f \wedge \alpha + f\bar{\partial}_E\alpha.$$

This correspondence is obviously given by $\bar{\partial}_E = L \cdot \bar{\partial}_{C(E)} \cdot L^{-1}$. If a bacs J is assigned on $C(E)$, one can define a linear differential operator $\bar{\partial}_E : \wedge^{p,q}(E) \to \wedge^{p,q+1}(E)$ in natural way, in fact, $\bar{\partial}_E = L \cdot \bar{\partial}_J \cdot L^{-1}$.

Proposition 2.2 ([3]) *The set $B(C(E))$ is in one-to-one correspondence with the set $\hat{H}(E)$.*

Definition 2.3 *Let $J \in B(C(E))$. Then a section e of E is said to be J-holomorphic if it satisfies $\bar{\partial}_E e = 0$, where the differential operator $\bar{\partial}_E$ is correspondence with J; this is equivalence to say that, if $\xi = L^{-1}(e) \in T^0(C(E))$, then $\bar{\partial}_J \xi = 0$.*

Definition 2.4 *Let $J \in B(C(E))$. Then a complex sub-bundle $E' \subset E$ is said to be a J-holomorphic subbundle if $\bar{\partial}_E$ maps $\wedge^{p,q}(E')$ into $\wedge^{p,q+1}(E')$.*

Definition 2.5 *Let $J \in B(C(E))$. A connection will be called type$(1,0)$, if it's connection 1-forms on $C(E)$ satisfies: $\omega \in T^{1,0}(C(E), gl(r,C), ad)$.*

Let $C_J^{1,0}(C(E))$ be the set of all connection 1-forms in $C(E)$ which are of type(1, 0) with respect to J. Given an $\omega \in C_J^{1,0}(C(E))$, it is easy to check that $D_\omega : T^0(C(E)) \to T^1(C(E))$ splits as $D_\omega = \partial_\omega + \bar{\partial}_J$, also we have the splitting $\nabla = \partial_\nabla + \bar{\partial}_E$ of the induced exterior covariant differential operator; and the $(1,1)$ part of curvature form is $F_\omega^{1,1} = \bar{\partial}_J \omega$ ([3] Proposition1.8; 1.9).

Assume a Hermitian metric H is assigned on E and let $U_H(E)$ be the principal $U(r)$-bundle of H-unitary frames on E, we have the following result:

Proposition 2.6 ([3]; proposition 2.1) *There exists a unique connection on $U_H(E)$ such that it's connection 1-form, when extended to a connection form on $C(E)$ is of type $(1,0)$ with respect to $J \in B(C(E))$; this connection is called the canonical Hermitian connection.*

Let $\hat{H} : C(E) \to GL(r,C)$ be defined as following: If $u = \{e_1, \dots e_r, \hat{J}e_1, \dots, \hat{J}e_r\}$, then $\hat{H}(u) = (H(e_j, e_k) - iH(e_j, \hat{J}e_k))_{1 \le j,k \le r}$. Set

$$\omega_H = \hat{H}^{-1} \partial_J \hat{H}, \tag{2.3}$$

it is just the canonical Hermitian connection 1-form correspondence with the metric structure H. Let K be another Hermitian structure on E and let $h = H^{-1}K$, it is easy to check that:

$$\omega_K = \omega_H + h^{-1} \partial_{\omega_H} h. \tag{2.4}$$

$$F_{\omega_K}^{1,1} = F_{\omega_H}^{1,1} + \bar{\partial}_E (h^{-1} \partial_{\omega_H} h). \tag{2.5}$$

We now suppose that the almost complex manifold M has a fixed Hermitian metric, with Kähler form η. The natural operator $\Lambda : \Omega_M^{1,1} \to \Omega_M^0$ is the contraction with η.

Choose a local real normal coordinate $(x^1, ..., x^{2m})$ centered at the considered point p_0. Let

$$J_M\left(\frac{\partial}{\partial x^\alpha}\right) = J_\alpha^\beta \frac{\partial}{\partial x^\beta}, \quad \alpha, \beta = 1, \dots, 2m.$$

By directly calculating, we have

$$-\sqrt{-1}\Lambda\bar{\partial}\partial f=\frac{1}{2}\triangle f+\frac{1}{2}\sum J_{\alpha}^{\beta}\frac{\partial J_{\beta}^{\gamma}}{\partial x^{\alpha}}\frac{\partial f}{\partial x^{\gamma}} \tag{2.6}$$

at the considered point p_0. Let $\tilde{\triangle}=-2\sqrt{-1}\Lambda\bar{\partial}\partial$, and $V=J_M(g^{\alpha\beta}(\nabla_{\frac{\partial}{\partial x^{\beta}}}J_M)(\frac{\partial}{\partial x^{\alpha}}))$, where $(g^{\alpha\beta})$ is the inverse matrix of the metric matrix in local coordinates. From the above equality, we have

$$\tilde{\triangle}f=\triangle f+\langle V,\nabla f\rangle, \tag{2.7}$$

for any $f\in C^2(M)$.

In the following parts, we will assume that complex vector bundle E has a fixed bundle almost complex structure (bacs) J, and ϕ be a nontrivial J-holomorphic section of E, then we consider the following generalized Yang-Mills-Higgs functional.

Definition 2.7 *We define the generalized Yang-Mills-Higgs functional*

$$YMH_{\lambda}:\mathcal{A}(H)\times\Omega^0(M,E)\to R$$

by

$$YMH_{\lambda}(A,\phi)=\|F_A\|_{L^2}^2+\|D_A\phi\|_{L^2}^2+\frac{1}{4}\|\phi\otimes\phi^*-\lambda Id\|_{L^2}^2. \tag{2.8}$$

Here, using the Hermitian metric H on E we get identifications $E\approx E^*$ and also $E\otimes E^*\approx End(E)$, $\mathcal{A}(H)$ denote connections on E that are compatible with H, ϕ^* is the adjoint of ϕ taken with respect to H and λ is a real parameter.

We have the following decomposition result for the above Yang-Mills-Higgs functional (also [2]).

Proposition 2.8 *Suppose that the Kähler form η satisfies $d\eta^{n-1}=0$, then the functional $YMH_{\lambda}:\mathcal{A}(H)\times\Omega^0(M,E)\to R$ can be written as*

$$\begin{aligned} YMH_{\lambda}(A,\phi)= \quad & 4\|F_A^{0,2}\|_{L^2}^2+2\|\bar{\partial}_A\phi\|_{L^2}^2+\|\sqrt{-1}\Lambda F_A^{1,1}+\tfrac{1}{2}\phi\otimes\phi^*-\tfrac{\lambda}{2}Id\|_{L^2}^2 \\ & +\lambda\int_M\sqrt{-1}Tr(F_A)\wedge\eta^{[n-1]}+\int_M Tr(F_A\wedge F_A)\wedge\eta^{[n-2]}. \end{aligned} \tag{2.9}$$

Here $\eta^{[m]}=\frac{\eta^m}{(m)!}$ and $F_A^{0,2}$ is the component of F_A of type $(0,2)$.

If M is an almost Kähler manifold (i.e. $d\eta=0$), the last two terms in (2.9) do not depend on the connection A. By Chern-Weil theorem, we known that they are determined by the first Chern class and second Chern character of E respectively. An immediate corollary is the following.

Corollary 2.9 When M is an almost Kähler manifold, the functional YMH_λ is bounded below by

$$2\pi\lambda C_1(E,\omega) - 8\pi^2 Ch_2(E,\omega),$$

and this lower bounded is attained at $(A,\phi) \in \mathtt{A}(H) \times \Omega^0(M,E)$ if and only if

$$F_A^{0,2} = 0, \tag{2.10}$$

$$\bar{\partial}_A \phi = 0, \tag{2.11}$$

$$\sqrt{-1}\Lambda F_A^{1,1} + \frac{1}{2}\phi \otimes \phi^* = \frac{\lambda}{2} Id. \tag{2.12}$$

The third equation generalizes the Hermitian-Yang-Mills equation (which is recovered by taking $\phi = 0$) and is the analgue of the classical Vortex equation over R^2. For this reasons we call the equation (2.12) the Hermitian Yang-Mills-Higgs or the Vortex equation.

The main purpose in this paper is to find a Hermitian metric H satisfying the Vortex equation

$$\Lambda F_H^{1,1} - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*H} + \lambda\frac{\sqrt{-1}}{2} Id = 0 \tag{2.13}$$

where F_H is the curvature form of the canonical metric connection ω_H. Let H_0 be a Hermitian metric on E. Consider a family of Hermitian metric $H(t)$ on E with initial metric $H(0) = H_0$. Denote by $\omega_{H(t)}$ and $F_{H(t)}$ the corresponding canonical metric connections and curvature forms, denote $h(t) = H_0^{-1}H(t)$. When there is no confusion, we will omit the parameter t and simply write H, ω_H, F_H, h for $H(t), \omega_{H(t)}, F_{H(t)}, h(t)$ respectively. The heat equation of (2.13) is

$$H^{-1}\frac{\partial H}{\partial t} = -2\sqrt{-1}(\Lambda F_H^{1,1} - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*H} + \frac{\sqrt{-1}}{2}\lambda Id). \tag{2.14}$$

It is completely equivalent to the following evolution equation

$$\frac{\partial h}{\partial t} = -2\sqrt{-1}\Lambda\bar{\partial}_E\partial_0 h + 2\sqrt{-1}\Lambda(\bar{\partial}_E h h^{-1}\partial_0 h) - \sqrt{-1}(\Lambda F_0 h + h\Lambda F_0) + \lambda h - h\phi \otimes \phi^{*H_0} h, \tag{2.15}$$

where $\partial_0 = \partial_{H_0}$. We know that the above equation is a nonlinear parabolic equation, as in [4], $h(t)$ are self adjoint with respect to H_0 for $t > 0$ since $h(0) = Id$. Discussing like that in [3], we have

Proposition 2.10 *Let $H(t)$ be a solution of heat flow (2.14), ϕ be a J-holomorphic section of E. Then*

$$(\frac{\partial}{\partial t} - \widetilde{\triangle})|\Lambda F_H^{1,1} - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*H} + \frac{\sqrt{-1}}{2}\lambda Id|_H^2 \leq 0. \tag{2.16}$$

Proof. By calculating directly, we have

$$\begin{array}{l}\frac{\partial}{\partial t}(\Lambda F_H^{1,1})=\frac{\partial}{\partial t}(\Lambda\bar{\partial}_E(h^{-1}\partial_0 h))\\ =\Lambda\bar{\partial}_E(\partial_H(h^{-1}\frac{\partial h}{\partial t}))\\ =-2\sqrt{-1}\Lambda\bar{\partial}_E(\partial_H(\Lambda F_H^{1,1}-\frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H}+\frac{\sqrt{-1}\lambda}{2}Id)).\end{array}\tag{2.17}$$

For simplicity, we denote $\Lambda F_H^{1,1}-\frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H}+\frac{\sqrt{-1}}{2}\lambda Id=\theta$. By calculating directly, we have

$$\begin{array}{rl}\widetilde{\triangle}|\theta|_H^2 & =-2\sqrt{-1}\Lambda\bar{\partial}\partial\{tr\theta H^{-1}\bar{\theta}^t H\}\\ & =-2\sqrt{-1}\Lambda tr\{\bar{\partial}\partial_H\theta H^{-1}\overline{\theta}^t H-\partial_H\theta H^{-1}\overline{\partial_H\theta}^t H+\bar{\partial}\theta H^{-1}\overline{\bar{\partial}\theta}^t H\}\\ & -2\sqrt{-1}\Lambda tr\{\theta H^{-1}\overline{\partial_H\bar{\partial}\theta}^t H\}\\ & =2Re\langle-2\sqrt{-1}\Lambda\bar{\partial}\partial_H\theta,\theta\rangle_H+Re\langle[2\sqrt{-1}\Lambda F_H^{1,1},\theta],\theta\rangle_H+2|\partial_H\theta|_H^2+2|\bar{\partial}_E\theta|_H^2.\end{array}\tag{2.18}$$

On the other hand, it is easy to check the following formula,

$$Re\langle\Lambda F_H^{1,1}\phi\otimes\phi^{*H}+\phi\otimes\phi^{*H}\Lambda F_H^{1,1},\Lambda F_H\rangle_H=2|\Lambda F_H\phi|_H^2,$$

$$\begin{array}{l}Re\langle\Lambda F_H^{1,1}\phi\otimes\phi^{*H}+\phi\otimes\phi^{*H}\Lambda F_H^{1,1},-\frac{\sqrt{-1}}{2}(\phi\otimes\phi^{*H}-\lambda Id)\rangle_H\\ =(|\phi|^2-\lambda)Re\langle-\sqrt{-1}\phi,\Lambda F_H^{1,1}\phi\rangle_H,\end{array}$$

$$Re\langle\sqrt{-1}\phi\otimes\phi^{*H}(\lambda Id-\phi\otimes\phi^{*H}),\Lambda F_H^{1,1}\rangle_H=(|\phi|^2-\lambda)Re\langle-\sqrt{-1}\phi,\Lambda F_H^{1,1}\phi\rangle_H,$$

$$Re\langle\sqrt{-1}\phi\otimes\phi^{*H}(\lambda Id-\phi\otimes\phi^{*H}),\frac{\sqrt{-1}}{2}(\lambda Id-\phi\otimes\phi^{*H})\rangle_H=\frac{1}{2}|\phi|_H^2(\lambda-|\phi|_H^2)^2.$$

Using above formulas, we have

$$\begin{array}{l}(\widetilde{\triangle}-\frac{\partial}{\partial t})|\Lambda F_H^{1,1}-\frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H}+\frac{\sqrt{-1}\lambda}{2}Id|_H^2\\ =2|\nabla_H(\Lambda F_H^{1,1}-\frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H}+\frac{\sqrt{-1}\lambda}{2}Id)|_H^2+\frac{1}{2}|-\sqrt{-1}(|\phi|^2-\lambda)\phi+2\Lambda F_H^{1,1}\phi|_H^2\\ \geq 0.\end{array}$$

□

Proposition 2.11 *Let $H(t)$ be a solution of heat flow (2.14) on $M\times[0,T)$, ϕ be a J-holomorphic section of E. Then*

$$(\frac{\partial}{\partial t}-\widetilde{\triangle})|\phi|_H^2=(\lambda-|\phi|_H^2)|\phi|_H^2-2|\partial_H\phi|_H^2.\tag{2.19}$$

Proof. By calculating directly, we have

$$\begin{array}{l}\frac{\partial}{\partial t}|\phi|_H^2=\frac{\partial}{\partial t}\langle\phi\otimes\phi^{*H},Id\rangle_H\\ =\langle\phi\otimes\phi^{*H}h^{-1}\frac{\partial h}{\partial t},Id\rangle_H+\langle\phi\otimes\phi^{*H}(2\sqrt{-1}\Lambda F_H^{1,1}),Id\rangle_H\\ -\langle(2\sqrt{-1}\Lambda F_H^{1,1})\phi\otimes\phi^{*H},Id\rangle_H\\ =\langle\phi\otimes\phi^{*H}(-\phi\otimes\phi^{*H}+\lambda Id)-2\sqrt{-1}\Lambda F_H^{1,1}\phi\otimes\phi^{*H},Id\rangle_H\end{array}$$

and

$$\widetilde{\triangle}|\phi|_H^2=2|\partial_H\phi|_H^2-2\langle\sqrt{-1}\Lambda F_H^{1,1}\phi,\phi\rangle_H,$$

where we have used $\bar{\partial}_E\phi = 0$. From above equalities we have

$$\begin{array}{ll} (\frac{\partial}{\partial t} - \widetilde{\triangle})|\phi|_H^2 & = -2|\partial_H\phi|^2 + \langle \phi\otimes\phi^{*H}(-\phi\otimes\phi^{*H} + \lambda Id), Id\rangle_H \\ & = -2|\partial_H\phi|^2 + (\lambda - |\phi|_H^2)|\phi|_H^2. \end{array}$$

□

Next, we will introduce the Donaldson's "distance" on the space of Hermitian metrics as follows.

Definition 2.12 *For any two Hermitian metrics H, K on bundle E set*

$$\sigma(H,K) = TrH^{-1}K + TrK^{-1}H - 2rankE. \tag{2.20}$$

It is obviously that $\sigma(H,K) \geq 0$ with equality if and only if $H = K$. The function σ is not quite a metric but it serves almost equally well in our problem. In particular, a sequence of metrics H_i converges to H in the usual C^0 topology if and only if $Sup_M\sigma(H_i, H) \longrightarrow 0$. Denoting $h = K^{-1}H$, applying $-\sqrt{-1}\Lambda$ to (2.5) and taking the trace in the bundle E, we have

$$Tr(\sqrt{-1}h(\Lambda F_H - \Lambda F_K)) = -\frac{1}{2}\widetilde{\triangle}Trh + Tr(-\sqrt{-1}\Lambda\bar{\partial}_E hh^{-1}\partial_K h). \tag{2.21}$$

Let $H(t)$, $K(t)$ be two solutions of Heat flow (2.14), Using the above formula, we have

$$(\widetilde{\triangle} - \frac{\partial}{\partial t})Trh(t) = 2Tr(-\sqrt{-1}\Lambda\bar{\partial}_E hh^{-1}\partial_K h) + Tr(h\phi\otimes\phi^{*H} - h\phi\otimes\phi^{*K}) \tag{2.22}$$

and

$$(\widetilde{\triangle} - \frac{\partial}{\partial t})Trh^{-1}(t) = 2Tr(-\sqrt{-1}\Lambda\bar{\partial}_E h^{-1}h\partial_K h^{-1}) + Tr(h^{-1}\phi\otimes\phi^{*K} - h^{-1}\phi\otimes\phi^{*H}). \tag{2.23}$$

Let $\{e_i\}$ be unitary basis with respect to metric K at the point under consideration, and suppose that $h(e_i) = \lambda_i e_i$.

$$\begin{array}{l} Tr(h\phi\otimes\phi^{*H} - h\phi\otimes\phi^{*K} + h^{-1}\phi\otimes\phi^{*K} - h^{-1}\phi\otimes\phi^{*H}) \\ = Tr((h - h^{-1})\phi\otimes\phi^{*K}(h - Id)) \\ = \sum_{i=1}^r \langle (h - h^{-1})\phi\otimes\phi^{*K}(h - Id)(e_i), e_i\rangle \\ = \sum_{i=1}^r (\lambda_i - 1)(\lambda_i - \lambda_i^{-1})|\langle\phi, e_i\rangle_K|^2 \\ = \sum_{i=1}^r (\lambda_i - 1)^2(\lambda_i + 1)(\lambda_i)^{-1}|\langle\phi, e_i\rangle_K|^2 \\ \geq 0. \end{array} \tag{2.24}$$

Using the above formula and the facts([4], [15])

$$Tr(-\sqrt{-1}\Lambda\bar{\partial}_E hh^{-1}\partial_K h) \geq 0, \quad Tr(-\sqrt{-1}\Lambda\bar{\partial}_E kk^{-1}\partial_H k) \geq 0, \tag{2.25}$$

where $k = h^{-1} = H^{-1}K$. We have

$$\begin{array}{l}(\widetilde{\triangle}-\frac{\partial}{\partial t})(Trh(t)+Trh^{-1}(t))\\ =2Tr(-\sqrt{-1}\Lambda\bar{\partial}_E hh^{-1}\partial_K h)+2Tr(-\sqrt{-1}\Lambda\bar{\partial}_E h^{-1}h\partial_H h^{-1})\\ +Tr(h\phi\otimes\phi^{*H}-h\phi\otimes\phi^{*K}+h^{-1}\phi\otimes\phi^{*K}-h^{-1}\phi\otimes\phi^{*H})\\ \geq 0.\end{array}$$

So we have proved the following proposition.

Proposition 2.13 *Let $H(t)$, $K(t)$ be two solutions of heat flow (2.14), then*

$$(\widetilde{\triangle}-\frac{\partial}{\partial t})\sigma(H(t),K(t))\geq 0. \tag{2.26}$$

Corollary 2.14 *Let H and K be two Hermitian metrics satisfy the Vortex equation (2.13), then $\sigma(H,K)$ satisfies:*

$$\widetilde{\triangle}\sigma(H,K)\geq 0. \tag{2.27}$$

Proposition 2.15 *Let $H(x,t)$ be a solution of heat flow (2.14) with the initial metric H_0, then*

$$(\widetilde{\triangle}-\frac{\partial}{\partial t})\lg\{Tr(H_0^{-1}H)+Tr(H^{-1}H_0)\}\geq -|2\sqrt{-1}\Lambda F_{H_0}+\phi\otimes\phi^{*H_0}-\lambda Id|_{H_0}. \tag{2.28}$$

Proof. Let $h=H_0^{-1}H$, applying (2.14) and (2.21), we have

$$(\widetilde{\triangle}-\frac{\partial}{\partial t})Trh=Tr(2\sqrt{-1}h\Lambda F_{H_0}+h\phi\otimes\phi^{*H}-\lambda h)+2Tr(-\sqrt{-1}\Lambda\bar{\partial}_E hh^{-1}\partial_0 h). \tag{2.29}$$

$$\begin{array}{ll}(\widetilde{\triangle}-\frac{\partial}{\partial t})Trh^{-1} & =-Tr(2\sqrt{-1}h^{-1}\Lambda F_{H_0}+\phi\otimes\phi^{*H}h^{-1}-\lambda h^{-1})\\ & +2Tr(-\sqrt{-1}\Lambda\bar{\partial}_E h^{-1}h\partial_H h^{-1}).\end{array} \tag{2.30}$$

Direct calculation shows that ([15])

$$\begin{array}{l}2(Trh)^{-1}Tr(-\sqrt{-1}\Lambda\bar{\partial}_E hh^{-1}\partial_0 h)-(Trh)^{-2}|\nabla Trh|^2\geq 0,\\ 2(Trh^{-1})^{-1}Tr(-\sqrt{-1}\Lambda\bar{\partial}_E h^{-1}h\partial_H h^{-1})-(Trh^{-1})^{-2}|\nabla Trh^{-1}|^2\geq 0.\end{array} \tag{2.31}$$

From above two inequalities, it is easy to check

$$\begin{array}{l}(Trh+Trh^{-1})^{-1}\{-2\sqrt{-1}\Lambda\bar{\partial}_E hh^{-1}\partial_0 h-2\sqrt{-1}\Lambda\bar{\partial}_E h^{-1}h\partial_H h^{-1}\}\\ \geq (Trh+Trh^{-1})^{-2}|\nabla Trh+\nabla Trh^{-1}|^2.\end{array} \tag{2.32}$$

Then, we have

$$\begin{array}{l}(\widetilde{\triangle}-\frac{\partial}{\partial t})\lg\{Trh+Trh^{-1}\}\\ =(Trh+Trh^{-1})^{-1}(\widetilde{\triangle}-\frac{\partial}{\partial t})\{Trh+Trh^{-1}\}-(Trh+Trh^{-1})^{-2}|\nabla Trh+\nabla Trh^{-1}|^2\\ =(Trh+Trh^{-1})^{-1}Tr(2\sqrt{-1}h\Lambda F_{H_0}+h\phi\otimes\phi^{*H_0}-\lambda h)\\ -(Trh+Trh^{-1})^{-1}Tr(2\sqrt{-1}h^{-1}\Lambda F_{H_0}+\phi\otimes\phi^{*H_0}h^{-1}-\lambda h^{-1})\\ +(Trh+Trh^{-1})^{-1}(Tr(h-h^{-1})\phi\otimes\phi^{*H_0}(h-Id))\\ +(Trh+Trh^{-1})^{-1}\{-2\sqrt{-1}\Lambda\bar{\partial}_E hh^{-1}\partial_0 h-2\sqrt{-1}\Lambda\bar{\partial}_E h^{-1}h\partial_H h^{-1}\}\\ -(Trh+Trh^{-1})^{-2}|\nabla Trh+\nabla Trh^{-1}|^2\\ \geq -|2\sqrt{-1}\Lambda F_{H_0}+\phi\otimes\phi^{*H_0}-\lambda Id|_{H_0}\end{array}$$

where we have used formula (2.24) and (2.32).

Using (2.21), (2.31), (2.32), and discussing like that in the above proposition, we have

Proposition 2.16 *Let $H(x)$ and $H_0(x)$ are two Hermitian metric, then*

$$\begin{aligned}\widetilde{\triangle}\lg\{TrH_0^{-1}H+TrH^{-1}H_0\} &\geq -|2\sqrt{-1}\Lambda F_{H_0}+\phi\otimes\phi^{*H_0}-\lambda Id|_{H_0}\\ &\quad -|2\sqrt{-1}\Lambda F_H+\phi\otimes\phi^{*H}-\lambda Id|_H.\end{aligned}\tag{2.33}$$

Corollary 2.17 *Let H be an Hermitian metric satisfying the Vortex equation (2.13), and H_0 be a fixed Hermitian metric, then*

$$\widetilde{\triangle}\lg\{Tr(H_0^{-1}H)+Tr(H^{-1}H_0)\}\geq -|2\sqrt{-1}\Lambda F_{H_0}+\phi\otimes\phi^{*H_0}-\lambda Id|_{H_0}.\tag{2.34}$$

At the end of this section, we use the Moser-iteration to deduce the following mean-value inequality which will be used in the proof of main theorem. The major geometric-analytic property of M which we are going to use is the Sobolev inequality on geodesic ball B_R. Namely, for any $\psi\in C_0^\infty(B(R))$, there exists a constant C_s only dependent on the geometry of M around $B(R)$ such that

$$C_s(\int_{B(R)}\psi^{\frac{4m}{2m-2}})^{\frac{2m-2}{2m}}\leq\int_{B(R)}|\nabla\psi|^2.\tag{2.35}$$

Theorem 2.18 *Suppose that nonnegative function f satisfies*

$$\widetilde{\triangle}f\geq -B_1f,\tag{2.36}$$

where B_1 is a positive constant. Let $p>0$, then there exist constant B_2 depending only on B_1, p and M such that

$$\sup_{B(\frac{R}{2})}f\leq B_2(\int_{B(R)}f^p)^{\frac{1}{p}}.\tag{2.37}$$

Proof. Setting $0<r_2<r_1\leq R$, and let φ be the cut-off function

$$\varphi(x)=\begin{cases}1; & x\in B(r_2)\\ 0; & x\in B(R)\setminus B(r_1)\end{cases}\tag{2.38}$$

$0\leq\varphi(x)\leq 1$ and $|\nabla\varphi|\leq 2(r_1-r_2)^{-1}$.

Let $q\geq p>1$. Multiplying $f^{q-1}\varphi^2$ on both side of (2.36) and integrating by parts we have

$$\begin{aligned}(q-1)\int_{B(R)}f^{q-2}\varphi^2|\nabla f|^2 &\leq -2\int_{B(R)}\langle\nabla\varphi,\nabla f\rangle f^{q-1}\varphi+\int_{B(R)}\langle V,\nabla f\rangle f^{q-1}\varphi^2\\ &\quad +B_1\int_{B(R)}f^q\varphi^2.\end{aligned}\tag{2.39}$$

Using Schwartz inequality and Young inequality, we have

$$\int_{B(R)} |\nabla(f^{\frac{q}{2}}\varphi)|^2 \leq \frac{q}{q-2}\int_{B(R)}(|V|^2+B_1)f^q\varphi^2 + \int_{B(R)} f^q|\nabla\varphi|^2. \tag{2.40}$$

Applying the Sobolev inequality (4.23) to $f^{\frac{q}{2}}\varphi$, we get

$$(\int_{B_{r_2}} f^{q\frac{2m}{2m-2}})^{\frac{2m-2}{2m}} \leq C(M,p,B_1,|V|)(1+(r_1-r_2)^{-2})\int_{B(r_1)} f^q. \tag{2.41}$$

Then, the standard Moser-iteration argument deduce (2.37) for $p>2$. On the other hand a general argument in [12] (or [17, proposition 2.2]) shows that $p>0$ case follows from $p>2$.

□

Corollary 2.19 *If nonnegative function f satisfies*

$$\widetilde{\triangle} f \geq -B_3 \tag{2.42}$$

then there exists positive constants B_4, B_5 depending only on M and B_3 such that

$$\|f\|_\infty \leq B_4(\|f\|_1 + B_5). \tag{2.43}$$

Proof. Let $f' = f + B_3$, then we have $\widetilde{\triangle} f' \geq -f'$. Applying the mean value inequality (2.37) to f', we can easily conclude the inequality (2.43).

□

3 (J,ϕ,λ)-stability of complex vector bundle

Let (M, J_M, g) be a compact m-dimensional almost Hermitian manifold whose Kähler form η satisfies $\partial_M\bar{\partial}_M\eta^{m-1}=0$, and $(E,\hat{J})$ be a complex vector bundle of rank r over M. We assume that complex vector bundle E has a fixed bundle almost complex structure (bacs) J, and assume further that E has a nontrivial J-holomorphic global section ϕ, i.e., ϕ satisfies $\bar{\partial}_E\phi=0$. Let H be a Hermitian metric on E, then we define the degree of E as follow:

$$\partial eg(E) = \frac{\sqrt{-1}}{2\pi}\int_M (Tr\Lambda F_H^{1,1})\eta^{[m]}. \tag{3.1}$$

From the condition of Kähler form, we know that the above definition is independent of Hermitian metrics H. Let $E' \subset E$ be a complex subbundle, using the Hermitian Codazzi-Mainardi equation, we have the following proposition ([3], proposition 2.4):

Proposition 3.1 *Let $(E,\hat{J},H)$ be a Hermitian bundle, let $E' \subset E$ be a complex subbundle, and let $J \in B(C(E))$. Then the following facts are equivalent:*

(1), E' is a J-holomorphic subbundle;

(2), the orthogonal projection $\pi : E \longrightarrow E'$ satisfies

$$(Id - \pi) \circ \bar{\partial}_{E^* \otimes E} \pi = 0. \tag{3.2}$$

For further consideration, let us introduce the following class of objects $\mathsf{F}(J)$ ([3]): $E' \in \mathsf{F}(J)$ if and only if

(1), there exists a closed subset $\Sigma \subset M$ with $H_{2m-4}(\Sigma) < +\infty$, such that $E'|_{M \setminus \Sigma}$ is a J-holomorphic subbundle of $E|_{M \setminus \Sigma}$;

(2), for any $x \in \Sigma$, and any local J_M-holomorphic curve C through x not contained in Σ, $E'|_{C - \{x\}}$ extends to C as subbundle

where H_s denote the s-dimensional Hausdorff measure. If $E' \in \mathsf{F}(J)$, it is easy to see that the corresponding section π of $E^* \otimes E$ is in $L_1^2(End(E))$. So it is possible to define the degree of E' as follow([3], [14]):

$$\partial eg(E') := \frac{1}{2\pi} \int_M (\sqrt{-1} Tr \pi \Lambda F_H^{1,1} - |\bar{\partial}_{E^* \otimes E} \pi|^2) \eta^{[m]}, \tag{3.3}$$

and the slope, $\mu(E')$, is defined

$$\mu(E') = \frac{\partial eg(E')}{rank E'}, \tag{3.4}$$

where H is any Hermitian metric on E. By Codazzi-Mainardi equations, if E' is regular, it is easy to check that this definition coincides with the one given in (3.1).

Definition 3.2 Let (E, J, ϕ) be a complex vector bundle with a fixed bundle almost complex structure J and a nontrivial J-holomorphic section ϕ. Given a real number λ, we say that E is (J, ϕ, λ)-stable if the following two conditions hold:

(1), $\mu(E') < \frac{\lambda Vol(M)}{4\pi}$, for every $E' \in \mathsf{F}(J)$;

(2), $\frac{r\mu(E) - r'\mu(E')}{r - r'} > \frac{\lambda Vol(M)}{4\pi}$, for every $E' \in \mathsf{F}(J)$ with $0 < rank E' = r' < r$ such that $\phi \in E'$ almost everywhere.

Next, we will show that the (J, ϕ, λ)-stability is the necessary condition for the existence of solutions of the Vortex equation (2.13). In fact, we prove the following theorem.

Theorem 3.3 *Let (M, J_M, g) be a compact m-dimensional almost Hermitian manifold whose Kähler form η satisfies $\partial_M \bar{\partial}_M \eta^{m-1} = 0$, and $(E, \hat{J})$ be a complex vector bundle of rank r over M. Let $J \in B(C(E))$ be a fixed bundle almost complex structure, and ϕ be a nontrivial J-holomorphic global section of E. Suppose that for a given real number $\lambda > 0$, there exists a Hermitian metric H satisfying the vortex equation*

$$\sqrt{-1} \Lambda F_H^{1,1} + \frac{1}{2} \phi \otimes \phi^{*H} - \frac{1}{2} \lambda Id = 0. \tag{3.5}$$

Then either

(a), (E,J,ϕ) is (J,ϕ,λ)-stable,

or

(b), the bundle splits as $E=E_\phi\otimes E'$ into a direct sum of J-holomorphic subbundles, one of which contains the section ϕ.

Proof: Let $E'\in \mathsf{F}(J)$ with $0<rankE'=r'<r$, and let π be the corresponding section of $E^*\otimes E$. Using (3.5), we have

$$\begin{aligned}\partial eg(E') &= \tfrac{1}{2\pi}\int_M(\sqrt{-1}Tr\pi\Lambda F_H^{1,1}-|\bar\partial\pi|^2)\eta^{[m]}\\ &= -\tfrac{1}{4\pi}\int_M|\pi\phi|_H^2-\tfrac{1}{2\pi}\int_M|\bar\partial\pi|^2+\tfrac{1}{4\pi}r'\lambda Vol(M),\end{aligned}$$

and so

$$\frac{\lambda Vol(M)}{4\pi}=\mu(E')+\frac{1}{4\pi r'}\int_M|\pi\phi|_H^2+\frac{1}{2\pi r'}\int_M|\bar\partial\pi|^2. \tag{3.6}$$

Consequently, it follows that $\mu(E')<\frac{\lambda Vol(M)}{4\pi}$, or $\int_M|\bar\partial\pi|^2$ and $\int_M|\pi\phi|_H^2$ are both zero. In the latter case, we know that π satisfying $D_H\pi=0$, so π is global regular and E splits $E=E'\otimes E^\perp$ J-holomorphically and ϕ is a section on $E^\perp$.

If $\phi\in E'$ almost everywhere, directly calculating, we have

$$\frac{r\mu(E)-r'\mu(E)}{r-r'}=\frac{1}{2\pi(r-r')}\int_M|\bar\partial\pi|^2+\frac{\lambda}{4\pi}Vol(M).$$

It follows that $\frac{r\mu(E)-r'\mu(E)}{r-r'}>\frac{\lambda Vol(M)}{4\pi}$, or E splits $E=E'\otimes E^\perp$. So we have proved this theorem. □

4 Proof of main theorem

In this section we will give a proof of main theorem using the heat equation method. Let (M,J_M,g) be a compact m-dimensional almost Hermitian manifold whose kähler form η satisfies $\partial_M\bar\partial_M\eta^{m-1}=0$, and $(E,\hat J)$ be a complex vector bundle of rank r over M. We assume that complex vector bundle E has a fixed bundle almost complex structure (bacs) J, and assume further that E has a nontrivial J-holomorphic global section ϕ. Let H_0 be the initial Hermitian metric on E, then we consider the following evolution equation

$$\left\{\begin{array}{l} H^{-1}\frac{d}{dt}H=-2(\sqrt{-1}\Lambda F_H^{1,1}+\frac12\phi\otimes\phi^{*H}-\frac12\lambda Id)\\ H(t)|_{t=0}=H_0.\end{array}\right. \tag{4.1}$$

Firstly, we will prove that the above equation has a long-time solution; nextly, under the assumption of stability, we will show that the solution converge to a Hermitian metric which we need. The main points in the discussion are similar with that in [3].

From formula (2.15), we know that the above equation is a nonlinear strictly parabolic equation, so standard parabolic theory gives the short-time existence.

Proposition 4.1 *For sufficiently small $\epsilon > 0$, the equation (4.1) have a smooth solution defined for $0 \le t < \epsilon$.*

Let $H(t)$ be a solution of the evolution equation (4.1), and $h(t) = H_0^{-1}H(t)$, then

$$\begin{aligned} |\frac{\partial}{\partial t}(\lg Trh)| &= |\frac{Tr(\frac{\partial h}{\partial t})}{Trh}| \\ &= |\frac{Trh(-2\sqrt{-1}\Lambda F_H^{1,1} - \phi\otimes\phi^{*H} + \lambda Id)}{Trh}| \\ &\le 2|\Lambda F_H^{1,1} - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \frac{\lambda\sqrt{-1}}{2}Id|_H, \end{aligned} \tag{4.2}$$

and similarly

$$|\frac{\partial}{\partial t}(\lg Trh^{-1})| \le 2|\Lambda F_H^{1,1} - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \frac{\lambda\sqrt{-1}}{2}Id|_H. \tag{4.3}$$

Theorem 4.2 *Suppose that a smooth solution H_t to the evolution equation (4.1) is defined for $0 \le t < T$. Then H_t converge in C^0 to some continuous non-degenerate metric H_T as $t \to T$.*

Proof: Given $\epsilon > 0$, by continuity at $t = 0$ we can find a δ such that

$$\sup_M \sigma(H_t, H_{t'}) < \epsilon,$$

for $0 < t, t' < \delta$. Then Proposition 2.13 and Maximum principle imply that

$$\sup_M \sigma(H_t, H_{t'}) < \epsilon,$$

for all $t, t' > T - \delta$. This implies that H_t are uniformly Cauchy sequence and converge to a continuous limiting metric H_T. On the other hand, by proposition 2.10, we know that $|\Lambda F_H^{1,1} - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \frac{\lambda\sqrt{-1}}{2}Id|_H$ are bounded uniformly. Using formula (4.2) and (4.3), one can conclude that $\sigma(H, H_0)$ are bounded uniformly, therefore $H(T)$ is a non-degenerate metric.

□

Discussing like that in [4; Lemma 19] or [8; Lemma 4.3.2], one can easily prove the following lemma.

Lemma 4.3 *Let $H(t)$, $0 \le t < T$, be any one-parameter family of Hermitian metrics on complex vector bundle E over almost Hermitian manifold M. If $H(t)$ converges in C^0 topology to some continuous metric H_T as $t \to T$, and if $\sup_M |\Lambda F_H^{1,1}|$ is bounded uniformly in t, then $H(t)$ are bounded in $C^{1,\alpha}$ (for $0 < \alpha < 1$) and also bounded in L_2^p (for any $1 < p < \infty$) uniformly in t.*

Theorem 4.4 *Given any initial Hermitian metric H_0, then the evolution equation (4.1) has a unique solution $H(t)$ which exists for $0 \leq t < \infty$.*

Proof. Proposition 4.1 guarantees that a solution exists for a short time. Suppose that the solution $H(t)$ exists for $0 \leq t < T$. By theorem 4.2, $H(t)$ converges in C^0 to a non-degenerate continuous limit metric $H(T)$ as $t \to t$. From proposition 2.10 and the maximum principle, we conclude that $|\Lambda F_H^{1,1} - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*H} + \frac{\sqrt{-1}\lambda}{2} Id|_H$ is bounded independently of t. Moreover, from proposition 2.11, we have

$$(\frac{\partial}{\partial t} - \widetilde{\triangle})|\phi|_H^2 \leq (\lambda - |\phi|_H^2)|\phi|_H^2.$$

Assume that $|\phi|_H^2$ attains its maximum on $M \times [0, T)$ at the point (x_0, t_0) with $0 < t_0 < T$, $x_0 \in M$. If $|\phi|_H^2(x_0, t_0) > \lambda$, then

$$(\frac{\partial}{\partial t} - \widetilde{\triangle})|\phi|_H^2(x_0, t_0) < 0.$$

This is contradicted with the maximum principle of the heat operator. Then $|\phi|_H^2$ must attain its maximum point at $t = 0$. So we have

$$|\phi|_H^2 \leq \max\{\sup_M |\phi|_{H_0}^2, \lambda\}. \tag{4.4}$$

Moreover, $\sup_M |\Lambda F_H^{1,1}|_{H_0}^2$ is bounded independently of t. Hence by lemma 4.3, $H(t)$ are bounded in C^1 and also bounded in L_2^p (for any $1 < p < \infty$) uniformly in t. Since the evolution (2.15) is quadratic in the first derivative of h we can apply Hamilton's method [6] to deduce that $H(t) \to H(T)$ in C^∞, and the solution can be continued past T. Then the evolution equation (4.1) has a solution $H(t)$ define for all time.

On the other hand, suppose that $K(t)$ is another solution of equation (4.1), from proposition 2.13, we have

$$(\widetilde{\triangle} - \frac{\partial}{\partial t})\sigma(H(t), K(t)) \geq 0,$$

and $\sigma(H, K)|_{t=0} = 0$. By the maximum principle, we have

$$\sigma(H(t), K(t)) \equiv 0, i.e. H(t) \equiv K(t).$$

So we have proved the the uniqueness of the solution.

□

Next, we will use the stability and discuss like that in [3] to deduce the existence of special Hermitian metric which we need. We need to introduce some further machineries. ([2], [3])

Let $J \in B(C(E))$, and H be a fixed Hermitian metric on E. Let

$$S_H(E) = \{s \in \Omega^0(M, End(E))| \quad s^{*H} = s\}.$$

Given $\varphi \in C^{\infty}(R,R)$ and $s \in S_H(E)$. We define $\varphi(s)$ as follows. At each point x on M, chose $\{e_i\}_{i=1}^r$ be an unitary basis with respect to metric H, such that $s(e_i) = \lambda_i e_i$. Set:

$$\varphi(s)(e_i) = \varphi(\lambda_i) e_i. \tag{4.5}$$

Given $\Phi \in C^{\infty}(R \times R, R)$, $s \in S_H(E)$, $p \in \Omega^0(M, End(E))$. In a similar way, we define $\Phi[s](p)$ as follows. Let $\{e_i^*\}_{i=1}^r$ be the dual basis for $\{e_i\}_{i=1}^r$, then $p \in \Omega^0(M, End(E))$ can be written

$$p = \sum p_{ij} e_i^* \otimes e_j.$$

Set:

$$\Phi[s](p) = \sum \Phi(\lambda_i, \lambda_j) p_{ij} e_i^* \otimes e_j. \tag{4.6}$$

If $\varphi \in C^{\infty}(R,R)$, then we set

$$\delta\varphi(x,y) = \begin{cases} \frac{\varphi(x)-\varphi(y)}{x-y}, & if x \neq y; \\ \varphi'(x), & if x = y. \end{cases}$$

For any $s \in S_H(E)$, it is easy to check that

$$\delta\varphi[s](\bar{\partial}_E s) = \bar{\partial}_E \varphi(s). \tag{4.7}$$

Define Donaldson's functional $V_H : S_H(E) \longrightarrow R$ as follows:

$$V_H(s) = 2\int_M \langle s, \sqrt{-1}\Lambda F_H^{1,1}\rangle_H + 2\int_M \langle s, \sqrt{-1}\Lambda\bar{\partial}_E(\Phi[s](\partial_H s))\rangle_H. \tag{4.8}$$

Let ϕ be a prescribed section of E and let λ be a real number. We define a real value functional as follows:

$$V_{\phi,\lambda,H}(s) = V_H(s) + \int_M \langle (e^s - Id)\phi, \phi\rangle_H - \lambda \int_M Trs, \tag{4.9}$$

and denote

$$M_{\phi,\lambda}(H,K) = V_{\phi,\lambda,H}(\log H^{-1}K), \tag{4.10}$$

where H and K are Hermitian metrics on E.

Lemma 4.5 *(1), Given $r \in S_H(E)$ and $s \in S_{He^r}(E)$, we have:*

$$V_{\phi,\lambda,H}(\log e^r e^s) = V_{\phi,\lambda,H}(r) + V_{\phi,\lambda,He^r}(s); \tag{4.11}$$

equivalently, if H_1, H_2, H_3 are three Hermitian metrics on E, then

$$M_{\phi,\lambda}(H_1,H_3) = M_{\phi,\lambda}(H_1,H_2) + M_{\phi,\lambda}(H_2,H_3). \tag{4.12}$$

(2), Let $H(t) = H_0 h(t) = H_0 e^{s(t)}$ be a family of Hermitian metrics on E, then

$$\frac{d}{dt} M_{\phi,\lambda}(H_0, H(t)) = \int_M \langle H^{-1}\frac{dH}{dt}, 2\sqrt{-1}\Lambda F_H^{1,1} + \phi\otimes\phi^* - \lambda Id\rangle_H. \tag{4.13}$$

Proof. (1), It is easy to check that

$$\langle (e^r e^s - Id)\phi, \phi\rangle_H = \langle (e^r - Id)\phi, \phi\rangle_H + \langle (e^s - Id)\phi, \phi\rangle_{He^r}, \tag{4.14}$$

and

$$Tr(\log e^r e^s) = Tr(r) + Tr(s). \tag{4.15}$$

On the other hand, the functional V_H satisfies the formula (4.11) ([3], lemma 3.6), and so $V_{\phi,\lambda,H}$ clearly does too.

(2), Using formula (4.12), we only need to compute $\frac{d}{dt}M_{\phi,\lambda}(H_0, H(t))$ for $t = 0$. It is easy to check that

$$\begin{aligned}\frac{d}{dt}M_{\phi,\lambda}(H_0, H(t))|_{t=t_0} &= \frac{d}{dl}M_{\phi,\lambda}(H_0, H(t_0 + l))|_{l=0}\\ &= \frac{d}{dl}M_{\phi,\lambda}(H_{t_0}, H(t_0 + l))|_{l=0},\\ &= \int_M \langle H^{-1}\frac{dH}{dt}, 2\sqrt{-1}\Lambda F_H^{1,1} + \phi\otimes\phi^* - \lambda Id\rangle_H|_{t=t_0}.\end{aligned}$$

□

For further argument, we need the following proposition.

Proposition 4.6 ([3]; Theorem 0.2) *Let (M, J_M, g), (N, J_N, h) be two almost Hermitian manifolds with $dim_R M = 2m$, and assume there exists a bounded closed 2-form α on N such that $\alpha^{1,1} > 0$ uniformly. Let $\sigma : M \longrightarrow N$ be a L_1^2-weakly (J_M, J_N)-holomorphic map. Then there exists a closed subset $\Sigma \subset M$ with $H_{2m-4}(\Sigma) < +\infty$, such that σ is smooth on $M \setminus \Sigma$; moreover, for any $x \in \Sigma$, any local J_M-holomorphic curve C through x not contained in Σ, $\sigma|_{C-\{x\}}$ extends smoothly to C.*

Proof of main theorem: Let $H(t) = H_0 h(t) = H_0 e^{s(t)}$ be a solution of equation (4.1). By proposition 2.10, we known that $\sup_M |2\sqrt{-1}\Lambda F_H^{1,1} + \phi\otimes\phi^{*H} - \lambda Id|_H$ is bounded independently of t. From proposition 2.16, we have

$$\begin{aligned}\widetilde{\triangle}\lg\{Trh + Trh^{-1}\} &\geq -|2\sqrt{-1}\Lambda F_{H_0} + \phi\otimes\phi^{*H_0} - \lambda Id|_{H_0}\\ &\quad -|2\sqrt{-1}\Lambda F_H + \phi\otimes\phi^{*H} - \lambda Id|_H.\end{aligned} \tag{4.16}$$

Using Corollary 2.19, there exists two constants C_0 and C_1 such that

$$\|\lg\{Trh + Trh^{-1}\}\|_\infty \leq C_1(\int_M \lg\{Trh + Trh^{-1}\} + C_0). \tag{4.17}$$

On the other hand, one can check that

$$\lg\{\frac{1}{2r}(Trh + Trh^{-1})\} \leq |s|_{H_0} = |s|_H \leq r^{-\frac{1}{2}}\lg(Trh + Trh^{-1}). \tag{4.18}$$

So there exist constants $C_2 > 0$, $C_3 > 0$ such that, for every $t \in [0, +\infty)$, we have:

$$\|s(t)\|_\infty \leq C_2 + C_3\|s(t)\|_1. \tag{4.19}$$

Now, there are two possibilities:

(1), There exists $C_4 > 0$ such that, for every $t \in [0, +\infty)$,

$$\|s(t)\|_\infty < C_4.$$

(2), $\limsup_{t\to\infty} \|s(t)\|_1 = +\infty$.

Assume we are in case (1). Using the condition $\partial_M \bar{\partial}_M \eta^{m-1} = 0$, it is not hard to check that

$$\begin{array}{l} \int_M \langle s, \sqrt{-1}\Lambda\bar{\partial}_E(\Phi[s](\partial_H s))\rangle_H \eta^{[m]} \\ = \int_M \langle \Psi[s](\bar{\partial}_E s), \bar{\partial}_E s\rangle_H \eta^{[m]} - \sqrt{-1}\int_M TrsH^{-1}\overline{\Phi[s](\partial_H s)}^T H \wedge \partial\eta^{m-1} \\ = \int_M \langle \Psi[s](\bar{\partial}_E s), \bar{\partial}_E s\rangle_H \eta^{[m]} - \frac{1}{2}\sqrt{-1}\int_M \bar{\partial}(Trs^2)\wedge\partial\eta^{m-1} \\ = \int_M \langle \Psi[s](\bar{\partial}_E s), \bar{\partial}_E s\rangle_H \eta^{[m]}. \end{array} \tag{4.20}$$

where $\Psi(x,y) = \Phi(y,x) = \frac{e^{x-y}-(x-y)-1}{(x-y)^2}$. For any $s \in S_H(E)$, we have

$$\begin{array}{rl} M_{\phi,\lambda}(H_0, H) & \geq -\int_M |s||2\sqrt{-1}\Lambda F_H^{1,1} - \lambda Id|\eta^{[m]} + 2\int_M \langle \Psi[s](\bar{\partial}_E s), \bar{\partial}_E s\rangle_H \eta^{[m]} \\ & + \int_M (|\phi|_H^2 - |\phi|_{H_0}^2). \end{array} \tag{4.21}$$

From $\|s(t)\|_\infty < C_4$ for every $t \in [0, +\infty)$, it follows that $\Psi \geq C_5 > 0$ on the range of the $s(t)$'s; so that

$$\int_M \langle \Psi[s](\bar{\partial}_E s), \bar{\partial}_E s\rangle_H \eta^{[m]} \geq C_5 \|\bar{\partial}_E s\|_2^2. \tag{4.22}$$

On the other hand, from theorem 4.4, we known that $|\phi|_{H(t)}$ is bounded uniformly. Therefore, there exists $C_6 > 0$ such that, for every $t \in [0, +\infty)$

$$M_{\phi,\lambda}(H_0, H(t)) \geq -C_6. \tag{4.23}$$

From (4.13), we have

$$\frac{d}{dt} M_{\phi,\lambda}(H_0, H(t)) = -\int_M |2\sqrt{-1}\Lambda F_H^{1,1} + \phi\otimes\phi^{*H} - \lambda Id|_H^2. \tag{4.24}$$

By (4.21), (4.22), we known that $\|\bar{\partial}_E s\|_2$ and also $\|\bar{\partial}_E h\|_2$ are uniformly bounded. Thus it follows that, there exits a subsequences $t_j \to +\infty$, such that $h(t_j)$ weakly converges to h_∞ in L_1^2. By (4.23) and (4.24), we known that $2\sqrt{-1}\Lambda F_{H(j)}^{1,1} + \phi\otimes\phi^{*H(j)} - \lambda Id$ weakly converges to 0 in L^2. Then, Lemma 4.3 and the standard elliptic regularity implies that h_∞ is smooth and $H_\infty = H_0 h_\infty$ satisfies the vortex equation (2.13).

Assume, from now on, we are in case (2). In particular, we can choose a sequence $\{t_j\}_{j=1}^\infty$ such that: $t_j \to \infty$ and $\|s(t_j)\|_1 \to \infty$. Let $l_j = \|s(t_j)\|_1$ and $u_j = l_j^{-1} s(t_j)$, then

$$\|u_j\|_1 = 1 \quad and \quad \|u_j\|_\infty \leq C_7 \tag{4.25}$$

where C_7 is a positive constant.

From

$$l_j\langle\Psi[l_ju_j](\bar\partial_E u_j),\bar\partial_E u_j\rangle \geq \langle\Psi[u_j](\bar\partial_E u_j),\bar\partial_E u_j\rangle \tag{4.26}$$

and (4.21) (4.24), it follows that

$$\int_M \langle\Psi[u_j](\bar\partial_E u_j),\bar\partial_E u_j\rangle\eta^{[m]} \leq C_8.$$

Since $\Psi \geq C > 0$ on the range of the u_j's, we obtain

$$\|\bar\partial_E u_j\| \leq C_9. \tag{4.27}$$

Then, passing to a subsequence, u_j converges weakly to u_∞ in L_1^2; clearly, u_∞ is nontrivial.

Moreover,

$$\begin{array}{l}\int_M\langle u_\infty, 2\sqrt{-1}\Lambda F_{H_0}^{1,1}-\lambda Id\rangle + 2\int_M\langle\Psi[u_\infty](\bar\partial_E u_\infty),\bar\partial_E u_\infty\rangle\\ = \lim_{j\to\infty}(\int_M\langle u_j, 2\sqrt{-1}\Lambda F_{H_0}^{1,1}-\lambda Id\rangle + 2\int_M\langle\Psi[u_j](\bar\partial_E u_j),\bar\partial_E u_j\rangle)\\ \leq \lim_{j\to\infty} l_j^{-1}\{\int_M\langle s(j), 2\sqrt{-1}\Lambda F_{H_0}^{1,1}-\lambda Id\rangle + 2\int_M\langle\Psi[s(j)](\bar\partial_E s(j)),\bar\partial_E s(j)\rangle\\ +\int_M(|\phi|_{H(j)}^2-|\phi|_{H_0}^2)\}\\ \leq \lim_{j\to\infty} l_j^{-1}M_{\phi,\lambda}(H_0,H(t_j)) = 0.\end{array} \tag{4.28}$$

In the same manner, if $\Theta\in C^\infty(R\times R, R)$ satisfies $\Theta(x,y)\leq(x-y)^{-1}$, whenever $x>y$, then

$$\int_M\langle u_\infty, 2\sqrt{-1}\Lambda F_{H_0}^{1,1}-\lambda Id\rangle + 2\int_M\langle\Theta[u_\infty](\bar\partial_E u_\infty),\bar\partial_E u_\infty\rangle \leq 0. \tag{4.29}$$

For any smooth function $f: R\to R$, by formula (4.7), we have

$$\bar\partial Trf(u_\infty) = Tr(\delta f[u_\infty](\bar\partial_E u_\infty)).$$

For any number N, we can find a smooth function $\tilde f: R\times R\to R$ such that: $\tilde f(x,x)=\delta f(x,x)$; and $N\tilde f^2(x,y)\leq(x-y)^{-1}$ whenever $x>y$. Then

$$\begin{aligned}|\bar\partial Tr(f(u_\infty))|^2 &= |Tr(\tilde f[u_\infty]\bar\partial_E u_\infty)| \leq C_9|\tilde f[u_\infty]\bar\partial_E u_\infty|^2\\ &= \tfrac{C_9}{N}\langle N\tilde f^2[u_\infty](\bar\partial_E u_\infty),\bar\partial_E u_\infty\rangle.\end{aligned}$$

By (4.29), we have

$$\|\bar\partial Tr(f(u_\infty))\|_2^2 \leq \frac{C_{10}}{N}. \tag{4.30}$$

Since this holds for all $N>0$, and $Tr(f(u_\infty))$ is real valued, we get that $Tr(f(u_\infty))$ is constant almost everywhere. This implies that the eigenvalues of u_∞ are constant almost everywhere, so we have proved the following lemma.

Lemma 4.7 *The eigenvalues of u_∞ are constant almost everywhere.*

Let $\lambda_1,\dots,\lambda_l$ denote the distinct eigenvalues of the u_∞, listed in ascending order. For $j<l$ define $p_j: R\to R$ to be a smooth positive function such that

$$p_j(x) = \begin{cases} 1. & if x\leq\lambda_j,\\ 0 & if x\geq\lambda_{j+1}\end{cases} \tag{4.31}$$

Define

$$\pi_j = p_j(u_\infty). \tag{4.32}$$

The same argument in [2; position 3.10.2] or [3; proposition 4.6] can deduce the following proposition.

Proposition 4.8 *Let π_j be as above for $j \leq l$. Then*
(1), $\pi_j \in L_1^2(S_{H_0}(E))$;
(2), $\pi_j^2 = \pi_j = \pi_j^{*H_0}$;
(3), $(Id - \pi_j)\bar{\partial}_{E^* \otimes E}(\pi_j) = 0$ almost everywhere;
Furthermore, if $\lambda_q \leq 0 < \lambda_{q+1}$, i.e. λ_q is the largest non-positive eigenvalue, then
(4), $\|(Id - \pi_q)\phi\|_2^2 = 0$ and
(5), not all the eigenvalues of u_∞ are positive.

From the above proposition, we known that π_j's are L_1^2-weakly J-holomorphic subbundles and correspond to L_1^2-weakly J-holomorphic maps from (M, J_M, g) to some Grassmann bundle $Gr_p(E)$. If $U \subset M$ is a sufficiently small domain, then π_{Gr}^{-1} can be equipped with a tamed Symplectic structure just by approximating the standard Kähler structure on $U \times Gr_p(C^r)$. Therefore proposition 4.6 implies that $\pi_j \in \mathsf{F}(J)$.

Using all the notation of the above, write

$$u_\infty = \lambda_l Id - \sum_{j=1}^{l-1}(\lambda_{j+1} - \lambda_j)\pi_j, \tag{4.33}$$

where $\hat{\lambda} = \frac{\lambda}{4\pi} Vol(M)$, and denote

$$Q(\hat{\lambda}) = rank(E)\lambda_l(\mu(E) - \hat{\lambda}) - \sum_{j=1}^{l-1}(\lambda_{j+1} - \lambda_j) rank(\pi_j)(\mu(\pi_j) - \hat{\lambda}). \tag{4.34}$$

Then

$$\begin{aligned} 2\pi Q(\hat{\lambda}) &= \textstyle\int_M \langle u_\infty, \sqrt{-1}\Lambda F_0^{1,1} - \frac{\lambda}{2} Id\rangle + \int_M \sum_{j=1}^{l-1}(\lambda_{j+1} - \lambda_j)|\bar{\partial}\pi_j|^2 \\ &= \textstyle\int_M \langle u_\infty, \sqrt{-1}\Lambda F_0^{1,1} - \frac{\lambda}{2} Id\rangle + \int_M \langle \sum_{j=1}^{l-1}(\lambda_{j+1} - \lambda_j)(\delta p_j)^2[u_\infty](\bar{\partial}u_\infty), \bar{\partial}u_\infty\rangle. \end{aligned} \tag{4.35}$$

Since $\sum_{j=1}^{l-1}(\lambda_{j+1} - \lambda_j)(\delta p_j)^2(x, y) \leq (x - y)^{-1}$ for $x > y$, it follows by (4.29) that

$$Q(\hat{\lambda}) \leq 0. \tag{4.36}$$

On the other hand, (J, ϕ, λ)-stability deduce that $Q(\hat{\lambda}) > 0$, so these conclude a contradiction. In fact, by proposition 4.8, $q \geq 1$, where λ_q is the largest non-positive eigenvalue of u_∞. So there are two cases to consider.

(a). Suppose $q = l$, i.e., all the eigenvalues are non-positive. In this case, we have that $\mu(E) - \hat{\lambda} < 0$ and $\mu(\pi_j) - \hat{\lambda} < 0$ for $j = 1, \ldots, l-1$. It follows immediately that

$$Q(\hat{\lambda}) = rank(E)\lambda_l(\mu(E) - \hat{\lambda}) - \sum_{j=1}^{l-1}(\lambda_{j+1} - \lambda_j)rank(\pi_j)(\mu(\pi_j) - \hat{\lambda}) > 0,$$

where we have used the fact that u_∞ is nontrivial.

(b). Suppose that $q < l$. By proposition 4.8, we have $\phi \in Im(\pi_q)$, and also $\phi \in Im(\pi_j)$ for $j \geq q$. The stability implies that

$$-rank(\pi_j)(\mu(\pi_j) - \hat{\lambda}) > -rank(E)(\mu(E) - \hat{\lambda}),$$

for $q \leq j < l$. Thus

$$\begin{aligned} Q(\hat{\lambda}) &= rank(E)\lambda_l(\mu(E) - \hat{\lambda}) - \textstyle\sum_{j=1}^{l-1}(\lambda_{j+1} - \lambda_j)rank(\pi_j)(\mu(\pi_j) - \hat{\lambda}) \\ &> \lambda_q r(\mu(E) - \hat{\lambda}) - \textstyle\sum_{j=1}^{q-1}(\lambda_{j+1} - \lambda_j)rank(\pi_j)(\mu(\pi_j) - \hat{\lambda}) \\ &> 0. \end{aligned}$$

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